N71-30062
NASA CR-118986

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

## Technical Memorandum 33-484

# Design of Subsystems in Large Structures

*T. K. Caughey*



JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

June 30, 1971

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

# Technical Memorandum 33-484

# Design of Subsystems in Large Structures

*T. K. Caughey*

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

June 30, 1971

Prepared Under Contract No. NAS 7-100
National Aeronautics and Space Administration

## PREFACE

The work described in this report was performed by the author while a consultant to the Jet Propulsion Laboratory, and under the cognizance of the Engineering Mechanics Division of the Jet Propulsion Laboratory. The author is Professor of Applied Mechanics at the California Institute of Technology.

CONTENTS

Abstract: A qualitative mathematical analysis is made of various approximate schemes which have been suggested in an attempt to short-cut the long and expensive computations which are required in the design of sub-systems in large, complex structures.

It is shown that the simple scheme of using the known blocked response of the main structure as the input to the sub-system does not, in general, lead to a convergent iterative process. The theory of the vibration absorber is used to illustrate this point.

A scheme which uses the response of a known system as the input to a new system, which differs only slightly from the known system, is shown to result in a convergent iterative process. It is shown that the first iterate of this process is sufficiently accurate for preliminary design purposes.

## Design of Sub-Systems in Large Structures.

Introduction: One of the major problems in the aerospace industry is the design of space craft for specified missions, given the structure of the booster and the dynamic excitation at various stages of the mission. This type of problem is typical of the class of problems encountered in the design of sub-systems in large complex systems. In such problems there is a strong interaction of the main structure and an interaction, not necessarily small, of the sub-system back on the main structure. In the past the design of such sub-systems has been tackled in a variety of ways ranging from cut and try methods all the way to formal analytical methods. At the present time the method most commonly used in the aerospace industry is to formulate a preliminary mathematical model of the space craft, mate this to the mathematical model of the booster and to solve this very large problem for each of the known excitations. Based on the results of this analysis a modified mathematical model of the space craft is formulated and the analysis is repeated. This process is repeated a number of times until a satisfactory design is achieved. Such a method is both time consuming and expensive in terms of computer time. The purpose of this report is to look into approximate methods by which time and expense might be saved, at least in the preliminary design phases.

### Formulation of the Problem:

Let S be a large structure having N degrees of freedom.

Let $S_1$ be the main structure having $N_1$ degrees of freedom.

Let $S_2$ be the sub-structure having $N_2=(N-N_1)$ degrees of freedom where in general $N_2<<N_1$.

Let

$$\underline{z} = \begin{Bmatrix} \underline{x} \\ \underline{\dot{x}} \end{Bmatrix} \text{ be the 2N state vector for S}$$

where

$$\underline{x} = \begin{Bmatrix} x_1 \\ \vdots \\ x_N \end{Bmatrix}$$

Let

$$\underline{z}_1 = \begin{Bmatrix} \underline{x}_1 \\ \underline{\dot{x}}_1 \end{Bmatrix} \text{ be the } 2N_1 \text{ state vector for } S_1$$

$$\underline{z}_2 = \begin{Bmatrix} \underline{x}_2 \\ \underline{\dot{x}}_2 \end{Bmatrix} \text{ be the } 2N_2 \text{ state vector for } S_2$$

Let

$$\underline{f}^*(t) = \begin{Bmatrix} \underline{0} \\ f(t) \end{Bmatrix} \text{ be the 2N "force" vector}$$

where $\underline{f}(t)$ is the N "force" vector applied to the structure. We shall assume that forces are applied only to $S_1$, therefore

$$\underline{f}(t) = \begin{Bmatrix} \underline{f}_1(t) \\ 0 \end{Bmatrix} \text{ where } f_1(t) \text{ is an } N_1 \text{ vector}$$

Equations of Motion: The equations of motion for the complete structure are

$$M\underline{\ddot{x}} + D\underline{\dot{x}} + K\underline{x} = \underline{f}(t) \tag{1}$$

In general M is symmetric and positive definite, however D and K, though symmetric, are in general only non-negative definite.

Equations (1) may be rewritten in a number of ways. We may split them up to reflect the structures $S_1$ and $S_2$ as follows

$$\left.\begin{aligned} M_1\ddot{\underline{x}}_1 + D_1\dot{\underline{x}}_1 + K_1\underline{x}_1 - (K_{12}\underline{x}_2 + D_{12}\dot{\underline{x}}_2) &= \underline{f}_1(t) \\ M_2\ddot{\underline{x}}_2 + D_2\dot{\underline{x}}_2 + K_2\underline{x}_2 - (K_{21}\underline{x}_1 + D_{21}\dot{\underline{x}}_1) &= 0 \end{aligned}\right\} \qquad (2)$$

where $(K_{12}\underline{x}_2 + D_{12}\dot{\underline{x}}_2)$ and $(K_{21}\underline{x}_1 + D_{21}\dot{\underline{x}}_1)$ represent the coupling between systems $S_1$ and $S_2$.

In terms of the state vectors $\underline{z}_1$ and $\underline{z}_2$ equations (2) may be written

$$\left.\begin{aligned} \frac{d\underline{z}_1}{dt} &= A_1\underline{z}_1 + B_{12}\underline{z}_2 + \underline{g}(t) \\ \frac{d\underline{z}_2}{dt} &= A_2\underline{z}_2 + B_{21}\underline{z}_1 \end{aligned}\right\} \qquad (3)$$

where

$$A_1 = \left[\begin{array}{c|c} 0 & I \\ \hline -M_1^{-1}K_1 & -M_1^{-1}D_1 \end{array}\right] \quad ; \quad \underline{g}(t) = \left\{\begin{array}{c} \underline{0} \\ \hline M_1^{-1}\underline{f}_1(t) \end{array}\right\} \quad ;$$

$$B_{12} = \left[\begin{array}{c|c} 0 & 0 \\ \hline M_1^{-1}K_{12} & M_1^{-1}D_{12} \end{array}\right] \quad ; \quad B_{21} = \left[\begin{array}{c|c} 0 & 0 \\ \hline M_2^{-1}K_{21} & M_2^{-1}D_{21} \end{array}\right] \quad ;$$

$$A_2 = \left[\begin{array}{c|c} 0 & I \\ \hline -M_2^{-1}K_2 & -M_2^{-1}D_2 \end{array}\right]$$

Equations (1) can also be expressed in terms of the state vector $\underline{z}$ for S.

$$\frac{d\underline{z}}{dt} = A\underline{z} + \underline{c}(t) \qquad (4)$$

where

$$A = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K & -M^{-1}D \end{array}\right] \quad ; \quad \underline{c}(t) = \left\{\begin{array}{c} \underline{0} \\ \hline M^{-1}\underline{f}(t) \end{array}\right\}$$

It should be pointed out that in most aerospace problems the matrices D and K in (1) are symmetric but only non-negative definite. If, however, the problem is formulated in terms of the relative displacement vector $\underline{y}$ thus

$$M\underline{\ddot{x}} + \overline{D}\underline{\dot{y}} + \overline{K}\underline{y} = \underline{f}(t) \tag{1'}$$

where

$$\underline{y} = R\underline{x}$$

If $(1)'$ is premultiplied by $RM^{-1}$ then $(1)'$ becomes

$$I\underline{\ddot{y}} + RM^{-1}\overline{D}\underline{\dot{y}} + RM^{-1}\overline{K}\underline{y} = RM^{-1}\underline{f}(t) \tag{2'}$$

where $RM^{-1}\overline{D}$ and $RM^{-1}\overline{K}$ are now positive definite.

If we define a new vector $\underline{z}^*$ by

$$\underline{z}^* = \begin{Bmatrix} \underline{y} \\ \underline{\dot{y}} \end{Bmatrix},$$

then $(2)'$ can be expressed as

$$\frac{d\underline{z}^*}{dt} = \overline{A}\underline{z}^* + \underline{c}^*(t) \tag{3'}$$

where

$$\overline{A} = \left[\begin{array}{c|c} 0 & I \\ \hline -RM^{-1}\overline{K} & -RM^{-1}\overline{D} \end{array}\right] \quad ; \quad \underline{c}^*(t) = \begin{Bmatrix} \underline{0} \\ RM^{-1}\underline{f}(t) \end{Bmatrix}$$

Comparison of the structure of equations $(3)'$ and (4) show that they are essentially the same, thus the discussion will be restricted to the case where M, D and K are symmetric and positive definite.

## Solution Techniques:

### Method 1. (Direct Solution)

From equations (4) we have immediately

$$\underline{z}(t) = e^{At}\underline{z}(0) + \int_0^t e^{A(t-\tau)}\underline{c}(\tau)d\tau \tag{5}$$

where $e^{At}$ is the principal matrix solution of

$$\frac{dX}{dt} = AX \quad ; \quad X(0) = I$$

and $\underline{z}(0)$ is the initial state vector.

In particular, if the initial conditions are zero, then

$$\underline{z}(t) = \int_0^t e^{A(t-\tau)} \underline{c}(\tau) d\tau \tag{6}$$

The forces in the system can be expressed as a linear function of the velocities and displacements in the system. Thus

$$\underline{f}_s(t) = F\underline{z} \tag{7}$$

where F is a $P \times 2N$ matrix with P is the number of structural elements in S.

If the excitation $\underline{c}(t)$ is a sample function of a stochastic process $\{\underline{c}(t)\}$ we may express the solution in terms of the mean value vector $\underline{m}_c(t)$ and the covariance matrix $Cov_c(t_1, t_2)$ of $\{\underline{c}(t)\}$. Where

$$\left.\begin{aligned} \underline{m}_c(t) &= E[\underline{c}(t)] \\ Cov_c(t_1, t_2) &= E\left[\left(\underline{c}(t_1) - \underline{m}_c(t_1)\right)\left(\underline{c}(t_2) - \underline{m}_c(t_2)\right)^T\right] \end{aligned}\right\} \tag{8}$$

Thus

$$\left.\begin{aligned} \underline{m}_z(t) &= E[\underline{z}(t)] = \int_0^t e^{A(t-\tau)} \underline{m}_c(\tau) d\tau \\ Cov_z(t_1, t_2) &= E\left[\left(\underline{z}(t_1) - \underline{m}_z(t_1)\right)\left(\underline{z}(t_2) - m_z(t_2)\right)^T\right] \\ &= \int_0^{t_1}\int_0^{t_2} e^{A(t_1-\tau_1)} Cov_c(\tau_1, \tau_2) e^{A^T(t_2-\tau_2)} d\tau_1 d\tau_2 \end{aligned}\right\} \tag{9}$$

The forces in the system can be expressed in terms of (9). Thus

$$\left.\underline{m}_{fs}(t) = E[f_s(t)] = Fm_z(t)\right\} \tag{10}$$

$$\left. \mathrm{Cov}_{fs}(t_1, t_2) = F\,\mathrm{Cov}_z(t_1, t_2) F^T \right\} \qquad \text{(10) cont.}$$

If the process of $\{\underline{c}(t)\}$ is jointly Gaussian then $\underline{f}_s(t)$ will also be Gaussian and (10) will enable us to calculate the probability that forces in the system will exceed any specified value. If the process $\{\underline{c}(t)\}$ is not Gaussian, (10) will still enable us to estimate the probability of exceeding any specified value.

If changes are made in the sub-structure all the calculations indicated above must be repeated, this is both time consuming and expensive. For this reason one is led to attempt approximate analysis techniques which will reduce the time and expense in preliminary design phases.

Method 2. (First Approximate Method)

Examination of equation (3) reveals that if the coupling term $B_{21}\underline{z}_1$ is known, then equations (3) can be integrated immediately to give

$$\underline{z}_2(t) = \int_0^t e^{A_2(t-\tau)} B_{21}\underline{z}_1(\tau)d\tau \qquad (11)$$

Where $\underline{z}_1(t)$ is the response of $S_1$ alone. Unfortunately, in general, the response of system $S_1$ depends on the response of system $S_2$, in fact from (3)

$$\left.\begin{aligned} \underline{z}_1(t) &= \int_0^t e^{A_1(t-\tau)} \underline{g}(\tau)d\tau + \int_0^t e^{A_1(t-\tau)} B_{12}\underline{z}_2(\tau)d\tau \\ \underline{z}_2(t) &= \int_0^t e^{A_2(t-\tau)} B_{21}\underline{z}_1(\tau)d\tau \end{aligned}\right\} \qquad (12)$$

Hence

$$\underline{z}_1(t) = \underline{z}_1^0(t) + \int_0^t e^{A_1(t-\tau)} B_{12}\int_0^\tau e^{A_2(\tau-\xi)} B_{21}\underline{z}_1(\xi)d\xi\, d\tau \qquad (13)$$

where

$$\underline{z}_1^0(t) = \int_0^t e^{A_1(t-\tau)} \underline{g}(t)d\tau$$

is the response of $S_1$ with $S_2$ blocked, i.e., $\underline{z}_2 \equiv 0$ and can be calculated with almost no knowledge of $S_2$.

Equation (13) is a linear integral equation of Volterra type; to solve it define a linear integral operator L, where

$$L\underline{\varphi} \equiv \int_0^t e^{A_2(t-\tau)} B_{12} \int_0^\tau e^{A_2(\tau-\xi)} B_{21}\underline{\varphi}(\xi)d\xi\, d\tau$$

The formal solution of (13) is easily obtained by the method of successive substitutions

$$\underline{z}_1(t) = \underline{z}_1^n(t) + L^{n+1}\underline{z}(t) \tag{14}$$

where

$$\underline{z}_1^n = \sum_{i=0}^{n} L^i \underline{z}_1^0(t)$$

If we allow n to tend to infinity, the formal solution is

$$\underline{z}_1(t) = \underline{z}_1^\infty(t)$$

The formal solution is said to exist for $\forall t$ if $\|\underline{z}_1(t)\| < \infty \, \forall t$. The symbol $\|\underline{z}_1(t)\|$ is the <u>norm</u> of $\underline{z}_1(t)$. We shall use the following norms

$$\|\underline{\varphi}\| = \sum_{i=1}^{N} |\varphi_i| \text{ for vectors}$$

$$\|C\| = \sum_i^N \sum_j^N |c_{ij}| \text{ for matrices}$$

Thus

$$\|L\underline{\varphi}\| = \left\|\int_0^t e^{A_1(t-\tau)} B_{12} \int_0^\tau e^{A_2(\tau-\xi)} B_{21}\underline{\varphi}(\xi)d\xi\, d\tau\right\|$$

$$\leq \int_0^t \|e^{A_1(t-\tau)} B_{12}\| \int_0^\tau \|e^{A_2(\tau-\xi)} B_{21}\| \; \|\underline{\varphi}(\xi)\| d\xi\, d\tau$$

$$\leq \left\{\int_0^t \|e^{A_1(t-\tau)} B_{12}\| \int_0^\tau \|e^{A_2(\tau-\xi)} B_{21}\| d\xi\, d\tau\right\} \operatorname{Sup}_t \|\underline{\varphi}(t)\|$$

Hence

$$\|L\underline{\varphi}\| \leq \|L\| \operatorname{Sup}_t \|\underline{\varphi}(t)\|$$

where

$$\|L\| \leq \int_0^t \|e^{A_1(t-\tau)} B_{12}\| \int_0^\tau \|e^{A_2(\tau-\xi)} B_{21}\| d\xi\, d\tau$$

Similarly it can be shown that

$$\|L^i\underline{\varphi}\| \leq \|L\|^i \operatorname{Sup}_t \|\underline{\varphi}(t)\|$$

Hence

$$\operatorname{Sup}_t \|\underline{z}_1(t)\| \leq \operatorname{Sup}_t \|\underline{z}_1^0(t)\| \sum_{i=0}^{\infty} \|L\|^i$$

$$\leq \operatorname{Sup}_t \|\underline{z}_1^0(t)\| \frac{1}{1 - \|L\|}$$

Thus if $\operatorname{Sup}\|\underline{z}_1^0(t)\| < \infty$, the final solution of (13) will exist if $\|L\| < 1$.

Now, $A_1$ and $A_2$ are stability matrices (by assumption), hence $\exists$ constants $K_1$, $K_2$, $\alpha_1$, $\alpha_2$ $\forall$ positive $\ni$

$$\|e^{A_1(t-\tau)} B_{12}\| \leq K_1 e^{-\alpha_1(t-\tau)} \quad , \quad \|e^{A_2(\tau-\xi)} B_{21}\| \leq K_2 e^{-\alpha_2(\tau-\xi)}$$

Hence

$$\|L\| \le \int_0^t K_1 K_2 e^{-\alpha_1(t-\tau)} \int_0^\tau e^{-\alpha_2(\tau-\xi)} d\xi \, dt$$

$$\le \frac{K_1 K_2}{\alpha_1 \alpha_2}$$

Thus a sufficient condition for the existence of a solution of (13) is that

$$\frac{K_1 K_2}{\alpha_1 \alpha_2} < 1$$

Returning to (14) we see that

$$\sup_t \|\underline{z}_1(t) - \underline{z}_1^n(t)\| \le \sup_t \|\underline{z}_1(t)\| \, \|L\|^{n+1}$$

$$\le \sup_t \|\underline{z}_1^0(t)\| \frac{\|L\|^{n+1}}{1 - \|L\|}$$

Hence

$$\left.\begin{aligned} \frac{\sup_t \|\underline{z}_1(t) - \underline{z}^n(t)\|}{\sup_t \|\underline{z}_1(t)\|} &\le \left(\frac{K_1 K_2}{\alpha_1 \alpha_2}\right)^{n+1} \\ \text{or} \quad \frac{\sup_t \|\underline{z}_1(t) - \underline{z}_1^n(t)\|}{\sup \|\underline{z}_1^0(t)\|} &\le \frac{\left(\frac{K_1 K_2}{\alpha_1 \alpha_2}\right)^{n+1}}{1 - \left(\frac{K_1 K_2}{\alpha_1 \alpha_2}\right)} \end{aligned}\right\} \tag{15}$$

Thus if

$$\frac{K_1 K_2}{\alpha_1 \alpha_2} < 1 \quad ; \quad \left(\frac{K_1 K_2}{\alpha_1 \alpha_2}\right)^{n+1} << 1$$

Hence

$$\underline{z}_1(t) \simeq \underline{z}^n(t) \tag{16}$$

Using (3)

$$\underline{z}_2(t) \simeq \int_0^t e^{A_2(t-\tau)} B_{21}\underline{z}_1^n(\tau)d\tau \tag{17}$$

In particular, if

$$\frac{K_1K_2}{\alpha_1\alpha_2} << 1$$

Then

$$\underline{z}_1(t) \simeq \underline{z}_1^0(t) \tag{18}$$

Hence

$$\underline{z}_2(t) \simeq \int_0^t e^{A_2(t-\tau)} B_{21}\underline{z}_1^0(\tau)d\tau \tag{19}$$

Since $\underline{z}_1^0(t)$ can be calculated from a knowledge of $S_1$ with little or no knowledge of $S_2$ since the dimension of $\dot{A}_2$ is much smaller than that of A or $A_1$, this looks like a very attractive method to reduce the amount of computing necessary for preliminary design. The difficulty is that the condition

$$\frac{K_1K_2}{\alpha_1\alpha_2} << 1$$

is a very restrictive condition which is unlikely to be satisfied in all but a few special cases, since $K_1 \simeq O(N_1^2)$, $K_2 \simeq O(N_2^2)$ and $\alpha_1$, $\alpha_2$ are measures of the smallest damping in structures $S_1$ and $S_2$. To illustrate this difficulty, this theory has been applied to the vibration absorber (see Appendix I) where it is shown that

$$\frac{\underset{t}{\mathrm{Sup}}\|\underline{z}_1(t)-\underline{z}_1^0(t)\|}{\mathrm{Sup}\|\underline{z}_2(t)\|} \leq \frac{2m}{M}\ \frac{1}{\zeta_1\zeta_2} \tag{20}$$

where m/M is the mass ratio and $\zeta_1$, $\zeta_2$ are respectively the damping ratios in $S_1$ and $S_2$. Thus it is seen that even if the mass ratio is small, the error is still very large if the product of the damping ratios is of the same order as the mass ratio.

While the conditions developed above are only sufficient conditions, the theory of the vibration absorber shows that if the sufficient conditions are violated large errors may indeed occur (see for example Mechanical Vibrations by Den Hartog).

Method 3 (Second Approximate Method)

From the discussion of Method 2 it might appear that there is little hope of developing an approximate technique which is sufficiently accurate for the purpose of preliminary design. This is probably true in general, however as we shall now show, there is a class of problems for which we can indeed develop a useful approximate method.

Suppose that we have a complete design of a prototype of the whole structure and wish to investigate the effect of small design changes in $S_2$ on the stresses and deformations in $S_2$. Let

$$\left.\begin{aligned} M_2 &= M_2^0 + \epsilon M_2^1 \\ K_2 &= K_2^0 + \epsilon K_2^1 \\ D_2 &= D_2^0 + \epsilon D_2^1 \end{aligned}\right\} \tag{21}$$

where $M_2^0, D_2^0, K_2^0$ reflect the prototype design of $S_2$, and $\epsilon M_2^1$, $\epsilon K_2^1$, $\epsilon D_2^1$ reflect the small design changes in $S_2$.

Equation (4) may be written in the form

$$\frac{d\underline{z}}{dt} = A\underline{z} + \epsilon B\underline{z} + \underline{c}(t) \tag{22}$$

where A represents the standard configuration and $\epsilon B$ reflects the changes in $S_2$.

It should be noted in passing that $\underline{c}(t)$ remains unchanged if the exciting forces are applied only to the main structure $S_1$. If the initial conditions are zero then (22) yields the linear integral equation

$$\underline{z}(t) = \int_0^t e^{A(t-\tau)}\underline{c}(\tau)d\tau + \epsilon\int_0^t e^{A(t-\tau)}B\underline{z}(\tau)d\tau \tag{23}$$

$$= \underline{z}^0(t) + \epsilon\int_0^t e^{A(t-\tau)}B\underline{z}(\tau)d\tau \tag{24}$$

where $\underline{z}^0(t)$ is the response of the standard configuration. Equation (24) is a regular Volterra integral equation of the second kind.

Let us define the linear integral operator L. Where

$$L\underline{\varphi}(t) = \epsilon\int_0^t e^{A(t-\tau)}B\underline{\varphi}(\tau)d\tau \tag{25}$$

The solution of (24) is obtained by the method of successive substitutions and is

$$\left.\begin{aligned} \underline{z}(t) &= \sum_{i=0}^{\infty} L^i\underline{z}^0(t) \\ \text{or}\quad \underline{z}(t) &= \sum_{i=0}^{n} L^i\underline{z}^0(t) + L^{n+1}\underline{z}(t) \end{aligned}\right\} \tag{26}$$

The solution exists if $\|L\| < 1$.

If $\|L\| < 1$, then error estimates can be made of the approximate solution $\underline{z}^n(t)$ where

$$\underline{z}^n(t) = \sum_{i=0}^{n} L^i\underline{z}^0(t)$$

Thus

$$\frac{\underset{t}{\mathrm{Sup}}\|\underline{z}(t)-\underline{z}^{n}(t)\|}{\underset{t}{\mathrm{Sup}}\|\underline{z}^{0}(t)\|} \leq \frac{\|L\|^{n+1}}{1-\|L\|} \tag{27}$$

or

$$\frac{\underset{t}{\mathrm{Sup}}\|\underline{z}(t)-\underline{z}^{n}(t)\|}{\underset{t}{\mathrm{Sup}}\|\underline{z}(t)\|} \leq \|L\|^{n+1} \tag{28}$$

Since A is a stability matrix $\exists$ constants K and $\alpha \ni$

$$\|e^{At}B\| \leq Ke^{-\alpha t} \quad , \quad \alpha > 0 \tag{29}$$

$$\therefore \quad \|L\underline{\varphi}\| \leq \frac{\epsilon K}{\alpha} \underset{t}{\mathrm{Sup}}\|\underline{\varphi}\| \tag{30}$$

Hence

$$\|L^{n}\underline{\varphi}\| \leq \left(\frac{\epsilon K}{\alpha}\right)^{n} \underset{t}{\mathrm{Sup}}\|\underline{\varphi}\| \tag{31}$$

Thus

$$\frac{\underset{t}{\mathrm{Sup}}\|\underline{z}(t)-\underline{z}^{n}(t)\|}{\underset{t}{\mathrm{Sup}}\|\underline{z}^{0}(t)\|} \leq \frac{\left(\frac{\epsilon K}{\alpha}\right)^{n+1}}{1-\left(\frac{\epsilon K}{\alpha}\right)} \tag{32}$$

$$\frac{\underset{t}{\mathrm{Sup}}\|\underline{z}(t)-\underline{z}^{n}(t)\|}{\underset{t}{\mathrm{Sup}}\|\underline{z}(t)\|} \leq \left(\frac{\epsilon K}{\alpha}\right)^{n+1} \tag{33}$$

Thus we can see that if $(\epsilon K/\alpha) < 1$

$$\underline{z}(t) \simeq \underline{z}^{n}(t) \tag{34}$$

In particular if $(\epsilon K/\alpha) << 1$

$$\underline{z}(t) \simeq \underline{z}_1(t) = \underline{z}^{0} + \epsilon\int_0^t e^{A(t-\tau)}B\underline{z}^{0}(\tau)d\tau \tag{35}$$

Thus an estimate of the displacements in the modified system can be obtained knowing the response of the standard system and the nature of the design changes in $S_2$. The forces in the system can be expressed as linear functions of the velocities and displacements. Thus

$$\underline{f}_s(t) = F\underline{z} \tag{36}$$

where F is a $P \times 2N$ matrix, P is the number of structural elements in S. Since

$$F = F^0 + \epsilon F' \tag{37}$$

where $F^0$ reflects the standard configuration and $\epsilon F'$ reflects the design changes. Thus

$$\begin{aligned} \underline{f}_s(t) = F\underline{z}(t) &\simeq (F^0 + \epsilon F')\left(\underline{z}^0(t) + \epsilon \int_0^t e^{A(t-\tau)} B\underline{z}^0(\tau) d\tau\right) \\ &= F^0 \underline{z}^0(t) + \left[\epsilon F^0 \int_0^t e^{A(t-\tau)} B\underline{z}^0(\tau) d\tau + F'\underline{z}^0\right] + O(\epsilon^2) \end{aligned} \tag{38}$$

Method 3 involves a great deal more computation than does Method 2, however, it has two great advantages:

i) The condition $\frac{\epsilon K}{\alpha} << 1$ is much more likely to be satisfied in practice than the condition

$$\frac{K_1 K_2}{\alpha_1 \alpha_2} << 1$$

ii) From (38) it is seen that $\underline{f}_s(t)$ is a linear function of the elements $B_{ij}$ of B, all other variables are known a priori. Thus, once the computations have been performed for the standard configuration, the results for <u>any</u> small design change can be obtained by simple matrix operations (see Appendix 2).

If $\underline{c}(t)$ is a member function of a stochastic process we can estimate the forces in the system in terms of the mean value vector $\underline{m}_c(t)$ and the covariance matrix $\mathrm{Cov}_c(t_1, t_2)$ of the process $\{\underline{c}(t)\}$, where $\underline{m}_c(t)$ and $\mathrm{Cov}_c(t_1, t_2)$ are defined in (8).

Using equation (26) with i=1, we have

$$\underline{m}_z(t) = E[\underline{z}(t)] \simeq E\ \underline{z}^0(t) + \epsilon\int_0^t e^{A(t-\tau)}B\underline{z}^0(\tau)d\tau \tag{39}$$

$$\therefore\ \underline{m}_z(t) \simeq \underline{m}_{z^0}(t) + \epsilon\int_0^t e^{A(t-\tau)}B\underline{m}_{z^0}(\tau)d\tau \tag{40}$$

where

$$\underline{m}_{z^0}(t) = \int_0^t e^{A(t-\tau)}\underline{m}_c(\tau)d\tau \tag{41}$$

$$\mathrm{Cov}_z(t_1, t_2) = E\left[\left(\underline{z}(t_1)-\underline{m}_z(t_1)\right)\left(\underline{z}(t_2)-\underline{m}_z(t_2)\right)^T\right] \tag{42}$$

$$\simeq \mathrm{Cov}_{z^0}(t_1, t_2) + \epsilon\left\{\int_0^{t_1} e^{A(t_1-\tau)} B\ \mathrm{Cov}_{z^0}(t_2, \tau)d\tau + \int_0^{t_2} \mathrm{Cov}_{z^0}(t_1, \tau)B^T e^{A^T(t_2-\tau)}d\tau\right\} + O(\epsilon^2) \tag{43}$$

where

$$\mathrm{Cov}_{z^0}(t_1, t_2) = \int_0^{t_1}\int_0^{t_2} e^{A(t_1-\tau_1)}\mathrm{Cov}_c(\tau_1, \tau_2)e^{A^T(t_2-\tau_2)}d\tau_1 d\tau_2 \tag{44}$$

The forces in the system can be expressed as linear functions of the velocities and displacements in the system. Thus

$$\underline{f}_s(t) = F\underline{z}(t) \tag{45}$$

Writing $F = F^0 + \epsilon F^1$, then

$$\underline{m}_{f_s}(t) = E[f_s(t)] \simeq F^0 \underline{m}_{z^0}(t) + \epsilon \left[ F' \underline{m}_{z^0}(t) + F^0 \int_0^t e^{A(t-\tau)} B m_{z^0}(\tau) d\tau \right] \tag{46}$$

Similarly

$$\begin{aligned} \mathrm{Cov}_{f_s}(t_1, t_2) = F^0 C_{z^0}(t_1, t_2) F^{0T} + \epsilon \Bigg[ & F' C_{z^0}(t_1, t_2) F^{0T} \\ & + F^0 C_{z^0}(t_1, t_2) F'^T + F^0 \Bigg\{ \int_0^{t_1} e^{A(t_1 - \tau)} B\, \mathrm{Cov}_{z^0}(t_2, \tau) d\tau \\ & + \int_0^{t_2} \mathrm{Cov}_{z^0}(t_1, \tau) B^T e^{A^T(t_2 - \tau)} d\tau \Bigg\} F^{0T} + O(t^2) \Bigg] \end{aligned} \tag{47}$$

It will be observed that (46) and (47) involve only simple matric operations on known quantities, thus if all the necessary quantities are calculated for the standard configuration then the second order statistics for <u>any</u> small design change can easily be determined.

Appendix I.

Application of Method 2 to the Vibration Absorber.

Consider the equations of motion for the vibration absorber

$$S_1 - M\ddot{x}_1 + B\dot{x}_1 + Kx_1 + k(x_1 - x_2) + \beta(\dot{x}_1 - \dot{x}_2) = f(t) \tag{1.1}$$

$$S_2 - m\ddot{x}_2 + \beta\dot{x}_2 + kx_2 = kx_1 + \beta\dot{x}_1 \tag{1.2}$$

Introducing the dimensionless time $\tau = \omega_0 t$ where

$$\omega_0 = \sqrt{\omega_1 \omega_2} \quad , \quad \omega_1^2 = \frac{K+k}{M} \quad , \quad \omega_2^2 = \frac{k}{m}$$

Let

$$\frac{\beta_1 + B}{M} = 2\omega_1 \zeta_1 \quad ; \quad \frac{\beta}{m} = 2\omega_2 \zeta_2$$

Then (1) and (2) become

$$x_1'' + 2\left(\frac{\omega_1}{\omega_0}\right)\zeta_1 x_1' + \left(\frac{\omega_1}{\omega_0}\right)^2 x_1 - \frac{1}{M}\left(\frac{k}{\omega_0^2}x_2 + \frac{\beta}{\omega_0}x_1'\right) = \frac{f(t)}{M} \tag{1.3}$$

$$x_2'' + 2\left(\frac{\omega_2}{\omega_0}\right)\zeta_2 x_2' + \left(\frac{\omega_1}{\omega_0}\right)^2 x_2 = \frac{1}{m}\left[\frac{k}{\omega_0^2}x_1 + \frac{\beta}{\omega_0}x_1'\right] \tag{1.4}$$

Thus

$$\left.\begin{aligned} \frac{d\underline{z}_1}{dt} &= A_1\underline{z}_1 + B_{12}\underline{z}_2 + \underline{f}(t) \\ \frac{d\underline{z}_2}{dt} &= A_2\underline{z}_2 + B_{21}\underline{z}_1 \end{aligned}\right\} \tag{1.5}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\left(\frac{\omega_1}{\omega_0}\right)^2 & -2\left(\frac{\omega_1}{\omega_0}\right)\zeta_1 \end{bmatrix} \quad ; \quad B_{12} = \frac{1}{M}\begin{bmatrix} 0 & 0 \\ \frac{k}{\omega_0^2} & \frac{\beta}{\omega_0} \end{bmatrix}$$

$$\underline{f(t)} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix} \quad ; \quad \underline{z}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -\left(\frac{\omega_2}{\omega_0}\right) & -2\left(\frac{\omega_2}{\omega_0}\right)\zeta_2 \end{bmatrix} \quad ; \quad B_{21} = \frac{1}{m}\begin{bmatrix} 0 & 0 \\ \frac{k}{\omega_0^2} & \frac{\beta}{\omega_0} \end{bmatrix}$$

$$z_2 = \begin{Bmatrix} x_2 \\ x_2' \end{Bmatrix}$$

Thus

$$\underline{z}_1 = \int_0^\tau e^{A_1(\tau-\xi)}\underline{f}(\xi)d\xi + \int_0^\tau e^{A_1(\tau-\xi)} B_{12}\int_0^\xi e^{A_2(\xi-\eta)} B_{12}\underline{z}_1(\eta)d\eta d\xi \tag{1.6}$$

To obtain $e^{A_1\tau}$, let $Z_1 = e^{A_1\tau}$, $Z_1$ satisfies

$$\frac{dZ_1}{dt} = A_1 Z_1 \quad \text{with } Z_1(0) = I \tag{1.7}$$

Now

$$Z_1 = \begin{bmatrix} x_1(\tau) & x_2(\tau) \\ x_1'(\tau) & x_2'(\tau) \end{bmatrix} \quad \text{with} \quad \begin{matrix} x_1(0) = x_2'(0) = 1 \\ x_1'(0) = x_2(0) = 0 \end{matrix}$$

Hence

$$Z_1(\tau) = e^{-(\omega_1/\omega_0)\zeta_1\tau}\begin{bmatrix} \left(\cos\theta + \frac{\zeta_1}{\sqrt{1-\zeta_1^2}}\sin\theta & \frac{1}{\left(\frac{\omega_1}{\omega_0}\right)}\frac{\sin\theta}{\sqrt{1-\zeta_1^2}} \\ -\frac{(\omega_1/\omega_0)\sin\theta}{\sqrt{1-\zeta_1^2}} & \cos\theta - \frac{\zeta_1}{\sqrt{1-\zeta_1^2}}\sin\theta \end{bmatrix} \tag{1.8}$$

where

$$\theta = \frac{\omega_2}{\omega_0}\sqrt{1-\zeta_1^2}\;\tau$$

Thus

$$e^{A_1(\tau-\xi)}B_{12} = \frac{1}{M}\begin{bmatrix} \frac{\frac{k}{2}}{\omega_0}x_2(\tau-\xi) & \frac{\beta}{\omega_0}x_2(\tau-\xi) \\ \frac{\frac{k}{2}}{\omega_0}x_2'(\tau-\xi) & \frac{\beta}{\omega_0}x_2'(\tau-\xi) \end{bmatrix} \tag{1.9}$$

$$\therefore \quad \|e^{A_1(\tau-\xi)}B_{12}\| = \frac{1}{M}\left[\frac{\frac{k}{2}}{\omega_0}+\frac{\beta}{\omega_0}\right]\left[|x_2'(\tau-\xi)| + |x_2(\tau-\xi)|\right]$$

Using the expressions for $x_2$ and $x_2'$

$$\|e^{A_1(\tau-\xi)}B_{12}\| \leq \frac{1}{M}\left[\frac{\frac{k}{2}}{\omega_0}+\frac{\beta}{\omega_0}\right]\sqrt{\left\{1+\frac{\left(1+\frac{\omega_1}{\omega_0}\zeta_1\right)^2}{\left(\frac{\omega_1}{\omega_0}\right)^2\left(1-\zeta_1^2\right)}\right\}}\, e^{-(\omega_1/\omega_0)\zeta_1(\tau-\xi)} \tag{1.10}$$

Similarly

$$\|e^{A_2(\xi-\eta)}B_{21}\| \leq \frac{1}{m}\left[\frac{\frac{k}{2}}{\omega_0}+\frac{\beta}{\omega_0}\right]\sqrt{\left\{1+\frac{\left(1+\frac{\omega_2}{\omega_0}\zeta_2\right)^2}{\left(\frac{\omega_2}{\omega_0}\right)^2\left(1-\zeta_2^2\right)}\right\}}\, e^{-(\omega_2/\omega_0)\zeta_2(\xi-\eta)} \tag{1.11}$$

Using (1. 10) and (1. 11) in (15) with $n = 0$

$$\frac{\underset{t}{\mathrm{Sup}}\|\underline{z}_1-\underline{z}_1^0\|}{\mathrm{Sup}\|\underline{z}_1\|} \leq \frac{1}{M_m}\,\frac{\left[\frac{\frac{k}{2}}{\omega_0}+\frac{\beta}{\omega_0}\right]^2}{\left(\frac{\omega_1}{\omega_0}\right)\left(\frac{\omega_2}{\omega_0}\right)\zeta_1\zeta_2}\sqrt{\left\{\left[1+\frac{\left(1+\frac{\omega_1}{\omega_0}\zeta_1\right)^2}{\left(\frac{\omega_1}{\omega_0}\right)^2\left(1-\zeta_1^2\right)}\right]\left[1+\frac{\left(1+\frac{\omega_2}{\omega_0}\zeta_2\right)^2}{\left(\frac{\omega_2}{\omega_0}\right)^2\left(1-\zeta_2^2\right)}\right]\right\}} \tag{1.12}$$

If

$$\frac{\omega_1}{\omega_0},\ \frac{\omega_2}{\omega_0} \simeq 1 \quad , \quad \zeta_1, \zeta_2 << 1 \quad , \quad k << K$$

Then

$$\frac{\sup_t \| \underline{z}_1 - \underline{z}_1^0 \|}{\sup \| \underline{z}_1 \|} \le \frac{2m}{M} \frac{1}{\zeta_1 \zeta_2}$$

We note that it is not enough that

$$\frac{m}{M} << 1$$

what is required is that

$$\frac{m}{M} << \zeta_1 \zeta_2$$

It is well known that if the damping is small, then a small mass ratio m/M can effect a drastic reduction of the main system $S_2$, particularly at "resonance" (Ref. 1).


Reference

1. "Mechanical Vibrations", J. P. DenHartog, pp. 108-127, Second Edition, McGraw-Hill Co., New York, 1940.

## Appendix II.

### Alternate Formulation of Method 3.

Instead of using the 2N space approach for Method 3, the problem can be solved in N space if it is assumed a priori that the standard configuration possesses classical normal modes. Using equation (21), equation (1) may be written

$$(M^0+\epsilon M^1)\ddot{\underline{x}}+(D^0+\epsilon D^1)\dot{\underline{x}}+(K^0+\epsilon K^1)\underline{x}=\underline{f}(t) \tag{2.1}$$

where $M^0, D^0, K^0$ reflect the standard configuration and $\epsilon M^1, \epsilon D^1, \epsilon K^1$ reflect the design changes in the system. Let

$$\underline{x}=\underline{x}^0+\epsilon\underline{x}^1+\epsilon^2\underline{x}^2+\ldots \tag{2.2}$$

Using standard perturbation techniques

$$\left.\begin{aligned}
&M^0\ddot{\underline{x}}^0+D^0\dot{\underline{x}}^0+K^0\underline{x}^0 = f(t) \quad \text{(i)}\\
&M^0\ddot{\underline{x}}^1+D^0\dot{\underline{x}}^1+K^0\underline{x}^1 = -[M^1\ddot{\underline{x}}^0+D^1\dot{\underline{x}}^0+K^1\underline{x}^0] \quad \text{(ii)}\\
&\text{-------------------------------------}\\
&M^0\ddot{\underline{x}}^{n+1}+D^0\dot{\underline{x}}^{n+1}+K^0\underline{x}^{n+1} = -[M^1\ddot{\underline{x}}^n+D^1\dot{\underline{x}}^n+K^1\underline{x}^n] \quad \text{(n+1)}
\end{aligned}\right\} \tag{2.3}$$

If it is assumed that $M^{0-1}D^0$ commutes with $M^{0-1}K^0$, then $\exists$ a nonsingular matrix T $\ni$

$$T^TM^0T=I \quad , \quad T^TD^0T=\left[\ddots\, 2\omega_n\zeta_n \ddots\right] \quad , \quad T^TK^0T=\left[\ddots\, \omega_n^2 \ddots\right] \tag{2.4}$$

Let

$$\underline{x}^0=T\underline{y}^0 \qquad T=[t_{ij}]$$

then

$$\ddot{y}_n+2\omega_n\zeta_n\dot{y}_n^0+\omega_n^2y_n^0=g_n(t) \tag{2.5}$$

where

$$g_n^0(t) = \sum_{j=1}^{N} t_{nj} f_j(t)$$

$$\therefore \quad y_n^0(t) = \int_0^t k_n(t-\tau) g_n(\tau) d\tau \tag{2.6}$$

$$k_n(t) = \frac{1}{\overline{\omega}_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

$$\overline{\omega}_n^2 = \omega_n^2 \left(1 - \zeta_n^2\right)$$

Similarly if we write

$$\underline{x}^k = T\underline{y}^k$$

then

$$\left.\begin{aligned} g_n^k(t) &= -\sum_{j=1}^{N} t_{nj} \left\{ M^1 \underline{\ddot{x}}^{k-1} + D^1 \underline{\dot{x}}^{k-1} + K^1 \underline{x}^{k-1} \right\}_j \\ y_n^k(t) &= \int_0^t k_n(t-\tau) g_n^k(\tau) d\tau \end{aligned}\right\} \tag{2.7}$$

Thus

$$\underline{x}(t) = T \sum_{k=0}^{\infty} \epsilon^k \underline{y}^k(t) \tag{2.8}$$

or

$$x_i(t) = \sum_{j=1}^{N} t_{ij} \sum_{k=0}^{\infty} \epsilon^k y_j^k(t) \tag{2.9}$$

In particular

$$x_i(t) \simeq x_i^0(t) + \epsilon \sum_{j=1}^{N} t_{ij} y_j^1(t) + O(\epsilon^2) \tag{2.10}$$

where

$$y_j^1 = \int_0^t k_j(t-\tau)\ {}_j^1(\tau)d\tau \tag{2.11}$$

$$g_j^1(t) = -\sum_{i=1}^{N} t_{ji}\left\{M^1\ddot{\underline{x}}^0 + D^1\dot{\underline{x}}^0 + K^1\underline{x}\right\}_i \tag{2.12}$$

$$= -\sum_{i=1}^{N} t_{ji} \sum_{k=1}^{N} m_{ik}^1\ddot{x}_k^0 + d_{ik}^1\dot{x}_k + h_{ik}^1 x_k^0 \tag{2.13}$$

Define

$$\left.\begin{aligned} \alpha_k^{(j)}(t) &= \int_0^t k_j(t-\tau)\ddot{x}_k^0(\tau)d\tau \\ \beta_k^{(j)}(t) &= \int_0^t k_j(t-\tau)\dot{x}_k^0(\tau)d\tau \\ \gamma_k^{(j)}(t) &= \int_0^t k_j(t-\tau)x_k^0(\tau)d\tau \end{aligned}\right\} \tag{2.14}$$

Then

$$y_j^1 = -\sum_{i=1}^{N}\ \sum_{k=1}^{N} t_{ji}\left[m_{ik}^1\alpha_k^{(j)}(t) + d_{ik}^1\beta_k^{(j)}(t) + k_{ik}^1\gamma_k^{(j)}(t)\right] \tag{2.15}$$

Hence

$$x_i(t) \simeq x_i^0(t) - \epsilon\sum_{j=1}^{N}\ \sum_{i=1}^{N}\ \sum_{k=1}^{N} t_{ij}t_{ji}\left[m_{ik}^1\alpha_k^{(j)}(t) + d_{ik}^1\beta_k^{(j)}(t) + k_{ik}^1\gamma_k^{(j)}(t)\right] + O(\varepsilon^2) \tag{2.16}$$

It is seen that the second term in (2. 16) is a linear function of the elements $m_{ik}^1$, $d_{ik}^1$, $k_{ik}^1$ which depend on the design changes, all other quantities are already known or can be calculated from the analysis of the standard configuration. Note also that i and k range over only those indices for which $m_{ik}^1$, $d_{ik}^1$, etc., are non-zero.


NASA — JPL — Coml., L.A., Calif.